JOURNAL OF DIFFERENTIAL EQUATIONS 31, 313–328 (1979)

Parametrices for a Class of Hypoelliptic Operators

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Received October 21, 1977; revised March 24, 1978

Using the general calculus of pseudodifferential operators developed by by Beals (*Duke Math. J.* **42** (1975), 1–42), parametrices are constructed for a class of degenerate semi-elliptic operators satisfying a quasi-homogeneity condition.

INTRODUCTION

In this paper we shall construct parametrices (inverses modulo smoothing operators) for the class of hypoelliptic operators considered by Taniguchi in [11]. The operators may be roughly described as degenerate semi-elliptic operators which satisfy a quasi-homogeneity condition. The operators considered by Grusin in [3] and [4] are included. Our methods also allow us to prove hypoellipticity for certain "variable coefficient" analogs of these operators and to show that hypoellipticity is preserved if they are perturbed by "lower order" terms (Theorem 1.13). These operators have received attention in recent years because they provide examples of variable coefficient hypoelliptic operators with multiple characteristics. Also the study of these operators parallels, to a certain extent, the study of hypoelliptic operators on nilpotent Lie groups. (See [10]). Our approach to these operators is an extension of that used by Beals in [2] for a few special cases. We shall make frequent use of the general pseudodifferential operator calculus developed in [1] and [2].

1. QUASI-HOMOGENEOUS SEMI-ELLIPTIC OPERATORS

We make the following conventions regarding notation: $\mathbb{N}$ is the set of non-negative integers, $\mathbb{Q}_+$ the non-negative rational numbers and $\mathbb{R}_+$ the non-negative real numbers. If $\rho, \alpha \in \mathbb{R}_+^n$, then

$$\alpha\rho = \sum \alpha_i \rho_i \qquad \text{and} \qquad | \alpha : \rho | = \sum \alpha_i \rho_i^{-1}.$$

0022-0396/79/030313-16$02.00/0

If $\rho \in \mathbb{R}_+^n$, $x \in \mathbb{R}^n$ and $t > 0$, then

$$t^\rho x = (t^{\rho_1} x_1, \ldots, t^{\rho_n} x_n).$$

D_j is always the operator $-i\partial/\partial x_j$. $\mathscr{D}$, $\mathscr{E}$ and $\mathscr{S}$ denote the usual spaces of test functions; $\mathscr{D}^*$, $\mathscr{E}^*$ and $\mathscr{S}^*$ their respective antilinear duals.

We now describe the operators to be considered. Let $m \in \mathbb{N}^n$, $m' \in \mathbb{N}^k$ and set $M = \max\{m_i, m'_j : 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k\}$. Let $\rho \in \mathbb{Q}_+^n$, $\rho' \in \mathbb{Q}_+^k$ and $\sigma' \in \mathbb{Q}_+^k$ be such that

(1.1) $\rho_i m_i = M$ for $i = 1, \ldots, n$;

(1.2) $\rho'_i m'_i \geqslant M$ for $i = 1, \ldots, k$;

(1.3) for some $s \leqslant k$, $\sigma'_i = 0$ if $s < i \leqslant k$;

(1.4) $\max\{\sigma'_i : i \leqslant s\} < \min\{m'_i \rho'_i / m'_j : 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant k\}$.

Note that this generalizes the situation in [3] and [4], where $m_i = m'_j = M$ for all i and j. In [3] s is always zero.

Let Λ be the subset of $\mathbb{N}^n \times \mathbb{N}^k \times \mathbb{N}^n \times \mathbb{N}^s$ such that $(\alpha, \beta, \gamma, \delta) \in \Lambda$ if and only if

$$|\alpha, \beta : m, m'| = |\alpha : m| + |\beta : m'| \leqslant 1, \tag{1.5}$$

and

$$\rho\alpha + \rho'\beta - \rho\gamma - \sigma'\delta = M. \tag{1.6}$$

Let $\Lambda_0 = \{(\alpha, \beta, \gamma, \delta) \in \Lambda : |\alpha, \beta : m, m'| = 1\}$. Let $a(x, y, \xi, \eta)$ be defined by

$$a(x, y, \xi, \eta) = \sum_{\Lambda} a_{\alpha\beta\gamma\delta} x^\gamma y^\delta \xi^\alpha \eta^\beta, \qquad \text{where each} \quad a_{\alpha\beta\gamma\delta} \in \mathbb{C}. \tag{1.7}$$

Condition (1.3) means that a is independent of $(y_{s+1}, \ldots, y_k)$ and condition (1.6) implies that a is quasi-homogeneous:

$$a(t^{-\rho} x, t^{-\sigma'} y, t^\rho \xi, t^{\rho'} \eta) = t^M a(x, y, \xi, \eta) \qquad \text{for} \quad t > 0. \tag{1.8}$$

Assume also that a satisfies the following two conditions:

(1.9) (Semi-ellipticity). If $(x; y_1, \ldots, y_s) \neq 0$ and $(\xi, \eta) \neq 0$, then

$$a_0(x, y, \xi, \eta) \neq 0,$$

where

$$a_0(x, y, \xi, \eta) = \sum_{\Lambda_0} a_{\alpha\beta\gamma\delta} x^\gamma y^\delta \xi^\alpha \eta^\beta.$$

(1.10) For all (y, η) such that $|\eta| = 1$, the equation $a(x, y, D_x, \eta)\, u = 0$ has no non-trivial solutions $u \in \mathscr{S}(\mathbb{R}^n)$.

We note that if a is given by (1.7) and satisfies (1.10) than the following stronger condition is also satisfied:

(1.11) For all (y, η) such that $|\eta| \neq 0$, the equation $a(x, y, D_x, \eta)\, u = 0$ has no non-trivial solutions $u \in \mathscr{S}(\mathbb{R}^n)$.

This follows from (1.8), for if $|\eta| \neq 0$, there exist $t \in \mathbb{R}_+$ and $\eta' \in \mathbb{R}^k$ such that $\eta' = t^{-\rho'}\eta$ and $|\eta'| = 1$. Define $V: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by $Vu(x) = u(t^\rho x)$. Then (1.8) implies

$$a(x, y, D_x, \eta) = t^M V a(x, t^{\sigma'} y, D_x, \eta')\, V^{-1},$$

so (1.11) follows from (1.10).

Definition. If P is a partial differential operator defined on an open set U, then $Q: \mathscr{E}^*(U) \to \mathscr{D}^*(U)$ is called a left *parametrix* for P on U if $QP - I$: $\mathscr{E}^*(U) \to \mathscr{E}(U)$. Q is said to be *very regular* if sing supp $Qv \subset$ sing supp v for all $v \in \mathscr{E}^*(U)$.

Clearly, if P has a very regular left parametrix, then P is hypoelliptic. The converse is an open problem. Our principal result is the following:

Theorem 1.12. *If a is given by* (1.7) *and satisfies* (1.9) *and* (1.10), *then the operator $A = a(x, y, D_x, D_y)$ has a very regular left parametrix on $\mathbb{R}^{n+k}$.*

The parametrix will be in the class of pseudodifferential operators described in Section 3. By the results of Taniguchi [11], if (1.9) is satisfied and $s = 0$, then (1.10) is also a necessary condition for hypoellipticity.

Theorem 1.13. *Let Λ' be the subset of $\mathbb{N}^{2(n+k)}$ such that $(\alpha, \beta, \gamma, \delta) \in \Lambda'$ if and only if* (1.5) *holds and*

$$\rho\alpha + \rho'\beta - \rho\gamma - \sigma'\delta \leqslant M.$$

For $(\alpha, \beta, \gamma, \delta) \in \Lambda'$ let $a'_{\alpha\beta\gamma\delta} \in C^\infty(U)$, where U is some open neighborhood of $(0, 0)$ in $\mathbb{R}^{n+k}$. Let

$$a'(x, y, \xi, \eta) = \sum_{\Lambda'} a'_{\alpha\beta\gamma\delta}(x, y)\, x^\gamma y^\delta \xi^\alpha \eta^\beta,$$

and let $a_{\alpha\beta\gamma\delta} = a'_{\alpha\beta\gamma\delta}(0, 0)$ for $(\alpha, \beta, \gamma, \delta) \in \Lambda$. Define a by (1.7). *If a satisfies* (1.9) *and* (1.10), *then $a'(x, y, D_x, D_y)$ is hypoelliptic on some neighborhood U_1 of* $(0, 0)$.

Both Theorem 1.12 and 1.13 will be proved in Section 3.

Definition. Let $p(x, y, \xi) = \sum_{|\alpha| \leqslant m} p_\alpha(x, y)\, \xi^\alpha$, where $(x, y) \in \Omega \subset \mathbb{R}_x^n \times \mathbb{R}_y^k$, $\xi \in \mathbb{R}^N$ and each p_α is smooth. Let $p^0(x, y, \xi) = \sum_{|\alpha| = m} p_\alpha(x, y)\, \xi^\alpha$. Then p is *strongly elliptic* on $\Omega \times \mathbb{R}^N$ if $\operatorname{Re} p^0(x, y, \xi) > 0$ on $\Omega \times (\mathbb{R}^N - \{0\})$.

COROLLARY 1.14. (See [7]) *Suppose that $p(x, y, \xi)$ is strongly elliptic on $\Omega \times \mathbb{R}^n$ and $q(x, y, \eta)$ is strongly elliptic on $\Omega \times \mathbb{R}^k$, where Ω is an open subset of $\mathbb{R}_x^n \times \mathbb{R}_y^k$. Then for any positive integer r, $A = p(x, y, D_x) + |x|^{2r} q(x, y, D_y)$ is hypoelliptic on Ω.*

Proof. If suffices to show that A is hypoelliptic on some neighborhood of each point of Ω. This is standard except when $|x| = 0$. By making a translation it suffices to show that A is hypoelliptic on some neighborhood of $(0, 0) \in \mathbb{R}^{n+k}$. We take $s = 0$. Let m be the order of p, m' that of q. Let $M = \max\{m, m'\}$. If $m \geqslant m'$, take $\rho_i = 1$ for $1 \leqslant i \leqslant n$ and $\rho_i' = (m + 2r)/m'$ for $1 \leqslant i \leqslant k$. If $m \leqslant m'$, take $\rho_i = m'm^{-1}$, $\rho_i' = (m + 2r)/m$. By Theorem 1.13, we may assume that both p and q have constant coefficients and are of principal type. We must verify that (1.10) is satisfied. Let $P = p(D_x)$. Suppose that

$$Pu + q(\eta) |x|^{2r} u = 0 \qquad \text{for} \quad u \in \mathscr{S}(\mathbb{R}^n).$$

Multiplying by $\bar{u}$ and integrating, we obtain

$$\operatorname{Re} \int Pu\bar{u}\, dx + \operatorname{Re} q(\eta) \int |x|^{2r} |u|^2\, dx = 0.$$

It follows from the strong ellipticity of p and q that $u = 0$.

COROLLARY 1.15 (See [8]). *Suppose that $q(x, y, \eta)$ is strongly elliptic of order m and has real coefficients on $\Omega \times \mathbb{R}^k$, where Ω is some open neighborhood of $(0, 0)$ in $\mathbb{R}_x^1 \times \mathbb{R}_y^k$. Then*

$$A = \frac{\partial}{\partial x} - cx^r q(x, y, D_y)$$

is hypoelliptic on Ω if $r > 0$ is even, $c \in \mathbb{R}$, and $c \neq 0$; or if $r > 0$ is odd and $c > 0$. Furthermore, hypoellipticity is preserved if A is perturbed by terms of the form $x^{r'}q'(x, y, D_y)$ where q' (not necessarily elliptic) has order $m' < m$ and $(r + 1)\, m' < (r' + 1)\, m$.

Proof. Let $\rho = m$ and $\rho_i' = r + 1$, for $1 \leqslant i \leqslant k$. Take s to be 0. Again by Theorem 1.13 we may assume that q has constant coefficients and is of principal type. The proof that (1.10) is satisfied for r even is the same as in the previous corollary. Suppose that r is odd, $u \in \mathscr{S}(\mathbb{R})$ and $u' - cq(\eta)\, x^r u = 0$, where $u' = du/dx$. Multiplying by $\bar{u}'$ and integrating yields

$$\| u' \|^2 - cq(\eta) \int x^r u\bar{u}'\, dx = 0,$$

hence $\int x^r u\bar{u}'\, dx \geqslant 0$, and taking the conjugate, $\int x^r \bar{u} u'\, dx \geqslant 0$. Adding these last two inequalities and integrating by parts yields $-\int x^{r-1} |u|^2\, dx \geqslant 0$. Hence $u = 0$.

Several special cases of these corollaries have been given elsewhere. For example, $(-\Delta_x)^m + |x|^{2r}(-\Delta_y)^m$ is considered in [3]. The operators $D_x \pm ix^r D_y{}^m$ on $\mathbb{R}_x{}^1 \times \mathbb{R}_y{}^1$ are considered in [5], [6] and [11]. For purposes of illustration we give two more examples where Theorems 1.12 and 1.13 apply.

EXAMPLE 1.16 ([4]). $a(x, y, \xi, \eta) = \xi^2 + (x^2 + y_1{}^2)(\eta_1{}^2 + \eta_2{}^2) + \lambda\eta_2$ on $\mathbb{R}_x{}^1 \times \mathbb{R}_y{}^2$. Here $s = 1$, $\rho = 1$, $\rho_2' = \rho_1' = 2$, $\sigma_1' = 1$. Condition (1.9) is satisfied for all λ. Consideration of the eigenvalues of $D_x{}^2 + x^2$ as an operator on $L^2(\mathbb{R}^1)$ shows that condition (1.10) is satisfied if $\operatorname{Im}\lambda \neq 0$, or $\operatorname{Im}\lambda = 0$ and $|\lambda| < 1$.

EXAMPLE 1.17. $a'(x, \xi, \eta) = \xi^4 + x^8\eta^4 + x^3\eta^2\xi + x^2\eta\xi^2 + \eta\xi$ on $\mathbb{R}_x{}^1 \times \mathbb{R}_y{}^1$. Here $\rho = 1$, $\rho' = 3$. Let $a(x, \xi, \eta) = \xi^4 + x^8\eta^4 + x^3\eta^2\xi + \eta\xi$. Then $a_0(x, \xi, \eta) = \xi^4 + x^8\eta^4$, so (1.9) is satisfied. Since

$$\operatorname{Im} \int (a(x, D_x, \eta)\, v(x))\, \overline{v(x)}\, dx = (3/2)\, \| xv \|^2 \qquad \text{if} \quad |\eta| = 1,$$

$a(x, D_x, \eta)\, v = 0$, $v \in \mathscr{S}$, implies $v = 0$. Hence a very regular left parametrix exists for $a(x, D_x, D_y)$ by Theorem 1.12, and $a'(x, D_x, D_y)$ is hypoelliptic by Theorem 1.13.

The theory of pseudodifferential operators that will be used in proving Theorems 1.12 and 1.13 depends on certain inequalities in which it will be necessary to replace the symbol a by a positive function g that dominates a but not by too much (see Lemma 1.22). First we introduce more notation. Given $m \in \mathbb{N}^n$ and $m' \in \mathbb{N}^k$, let ρ, ρ' and σ' satisfy (1.1)–(1.4). Define

$$|x, y|_{\rho,\sigma'} = \sum_{i=1}^{n} |x_i|^{1/\rho_i} + \sum_{i=1}^{s} |y_i|^{1/\sigma_i'}, \tag{1.18}$$

$$h(\eta) = \sum_{i=1}^{k} |\eta_i|^{1/\rho_i'} + 1, \tag{1.19}$$

$$g_0(x, y, \xi, \eta) = \sum_{i=1}^{n} |\xi_i|^{m_i} + \sum_{i=1}^{k} |x, y|_{\rho,\sigma'}^{m_i'\rho_i' - M} |\eta_i|^{m_i'}, \tag{1.20}$$

$$g(x, y, \xi, \eta) = g_0(x, y, \xi, \eta) + h(\eta)^M. \tag{1.21}$$

LEMMA 1.22. *If the symbol a of* (1.7) *satisfies* (1.9), *then there exists a C such that*

$$g_0(x, y, \xi, \eta) \leqslant C\, |a_0(x, y, \xi, \eta)|. \tag{1.23}$$

If $(\alpha, \beta, \gamma, \delta) \in \Lambda$, then $|x^\gamma y^\delta \xi^\alpha \eta^\beta| \leqslant g(x, y, \xi, \eta)$, so

$$|a(x, y, \xi, \eta)| \leqslant Cg(x, y, \xi, \eta) \qquad \textit{for all} \quad (x, y, \xi, \eta). \tag{1.24}$$

Furthermore, given $C_1 > 0$ *and* d, $0 < d < 1$, *there exists* $C > 0$ *and* $C' > 0$ *such that*

$$g(x, y, \xi, \eta) \leqslant C \mid a(x, y, \xi, \eta)\mid \textit{ if } g(x, y, \xi, \eta) \geqslant C' \textit{ and } h(\eta)^M \leqslant C_1 g(x, y, \xi, \eta)^d. \tag{1.25}$$

Proof. (1.23) follows easily from (1.9) and the fact that both g and a satisfy the quasi-homogeneity condition (1.8), (see [11]). The other assertions follow from the weighted geometric-arithmetic mean inequality:

$$\prod t_i^{\lambda_i} \leqslant \sum \lambda_i t_i \qquad \text{if } \sum \lambda_i = 1, \tag{1.26}$$

where all λ_i and t_i are positive. Details are given in [9]. Consideration of the symbol a of Example 1.17, in which case $g(x, \xi, \eta) \sim \xi^4 + x^8\eta^4 + \eta^{4/3} + 1$, will indicate what is involved.

2. Inverses for the Operators $a(x, y, D_x, \eta)$

Let a be the symbol of (1.7) and assume that (1.9) and (1.10) hold. The proof of Theorem 1.12 will make use of a lemma (Lemma 2.7) to the effect that the operators $a(x, y, D_x, \eta)$, with y and η considered as parameters, have inverses which are pseudodifferential operators. We will need to take special care to show that the symbols of these inverses satisfy certain estimates uniformly on the set $I = \{(y, \eta) \in \mathbb{R}^{2k}: c_1 \leqslant |\eta| \leqslant C_1\}$, where c_1 and C_1 are constants $0 < c_1 < C_1$. If p is any function of the variables x, y, ξ and η, we will let $p_{y\eta}(x, \xi) = p(x, y, \xi, \eta)$ when y and η are being considered as parameters.

Let g be the function defined by (1.21), and choose $\bar{\rho} \in \mathbb{R}$, $\bar{\rho} \geqslant \max\{\rho_i, \rho_j', 2\}$. For $i = 1, 2,..., n$, let

$$\begin{aligned} X_i(x, y, \xi, \eta) &= g(x, y, \xi, \eta)^{1/m_i}, \\ \chi_i(x, y, \xi, \eta) &= X_i(x, y, \xi, \eta)^{1/(\bar{\rho}-1)} + |x_i|, \\ \chi_i{}^0(x, y, \xi, \eta) &\equiv 1. \end{aligned} \tag{2.1}$$

Note that there is a constant $C > 0$ such that

$$g(x + \chi(x, y, \xi, \eta), y, \xi + X(x, y, \xi, \eta), \eta) \leqslant Cg(x, y, \xi, \eta). \tag{2.2}$$

Lemma 2.3. *For each* $(y, \eta) \in I$, $(X_{y\eta}, \chi_{y\eta}^0)$ *is a pair of strongly coercive weight vectors on* $\mathbb{R}^n$, *in the sense of* [2].

Proof. Each of the conditions (2.1)–(2.7) and (7.4) of [2] is trivially verified except (2.3), (2.5) and (2.7). Condition (2.3) of [2] follows from (2.2) above and the fact that $\chi_i{}^0 \leqslant \chi_i$. Conditions (2.5) and (2.7) of [2] follow from condition (2.3) of [2] in this case. Note that the constants c, C and δ in conditions (2.1)–(2.7) and (7.4) of [2] can be chosen independent of $(y, \eta) \in I$.

For weight vectors Φ and φ let $O(\Phi, \varphi)$ denote the set of orders as defined in [2].

DEFINITION. Let μ be a function of x, y, ξ and η such that for each $(y, \eta) \in I$, $\mu_{y\eta} \in O(X_{y\eta}, \chi^0_{y\eta})$, and the constants c, C, k, K and m in conditions (3.1) and (3.2) of [2] can be chosen independent of $(y, \eta) \in I$. Then we write $\mu \in O(X, \chi^0; I)$. If $\mu \in O(X, \chi^0; I)$, let $S_0{}^\mu(\mathbb{R}^n; I)$ be the set of all functions $p(x, y, \xi, \eta)$ defined for $(x, \xi) \in \mathbb{R}^{2n}$, $(y, \eta) \in I$, smooth in x and ξ for each $(y, \eta) \in I$, and such that for any multi-indices α, β

$$\sup e^{-\mu} X^\alpha (\chi^0)^\beta \mid D^{\alpha\beta}_{\xi x} p \mid < \infty, \tag{2.4}$$

where the supremum is taken over $(x, \xi) \in \mathbb{R}^{2n}$, $(y, \eta) \in I$.

Although $(X_{y\eta}, \chi_{y\eta})$ is not a pair of weight vectors as defined in [2], we shall denote by $S^\mu(\mathbb{R}^n; I)$ the set of all functions p, as above, for which (2.4) is satisfied with χ^0 replaced by χ. Note that $S^\mu(\mathbb{R}^n; I) \subset S_0{}^\mu(\mathbb{R}^n; I)$.

LEMMA 2.5. *Let μ and ν be in $O(X, \chi^0; I)$. If $p \in S_0{}^\mu(\mathbb{R}^n; I)$ and $q \in S_0{}^\nu(\mathbb{R}^n; I)$, then $p \circ q \in S_0^{\mu+\nu}(\mathbb{R}^n; I)$. If $p \in S^\mu(\mathbb{R}^n; I)$ and $q \in S^\nu(\mathbb{R}^n; I)$, then $p \circ q \in S^{\mu+\nu}(\mathbb{R}^n; I)$.* (In this section, $p \circ q$ denotes the symbol of $p(x, y, D_x, \eta)\, q(x, y, D_x, \eta)$.)

Proof. The first statement is the composition theorem, Theorem 4.1, of [2]. The fact that the required estimates (2.4) for $p \circ q$ are uniform for $(y, \eta) \in I$ can be seen by noting that in the proof of Theorem 4.1 all constants depend only on the constants c, C, k, K and m of conditions (2.1)–(2.7) and (3.1)–(3.2) of [2], and these constants are independent of $(y, \eta) \in I$.

To prove the second statement let $J_N = \{\alpha \in \mathbb{N}^n : \alpha_i \leqslant N m_i \text{ for all } i\}$ for N a positive integer. If $\alpha \notin J_N$, then

$$S_0^{\mu+\nu-(\alpha)}(\mathbb{R}^n; I) \subset S_0^{\mu+\nu-(N)}(\mathbb{R}^n; I)$$

where $(\alpha) = \sum \alpha_i \log X_i$ and $(N) = N \log g$. By Theorem 4.1 of [2], $p \circ q = s_N + r_N$, where $s_N = \sum_{\alpha \in J_N} (\alpha!)^{-1} D_\xi{}^\alpha p \partial_x{}^\alpha q$ and $r_N \in S_0^{\mu+\nu-(N)}(\mathbb{R}^n; I)$. For any $N \geqslant 0$, certainly $s_N \in S^{\mu+\nu}(\mathbb{R}^n; I)$. If N (depending on β) is sufficiently large, then

$$X^\alpha \chi^\beta \mid D^{\alpha\beta}_{\xi x} r_N \mid \leqslant C e^{\mu+\nu},$$

since $\chi_i \leqslant C g^K$ for some fixed $K > 0$. Thus $p \circ q \in S^{\mu+\nu}(\mathbb{R}^n; I)$.

LEMMA 2.6. *Let $\mu = \log g$. Then $\mu \in O(X, \chi^0; I)$, and $a \in S^\mu(\mathbb{R}^n; I)$.*

Proof. Since $g = X_1^{m_1}$, $\log g \in O(X, \chi^0; I)$. Now let $(\alpha', \beta', \gamma', \delta') \in \Lambda$, Λ

defined as at (1.5). By Lemma 1.22 $|x^{\gamma'}y^{\delta'}\xi^{\alpha'}\eta^{\beta'}| \leqslant g(x, y, \xi, \eta)$. Using (2.2) we obtain

$$(x+\chi)^{\gamma'}y^{\delta'}(\xi+X)^{\alpha'}\eta^{\beta'} \leqslant Cg.$$

Hence $\chi^{\gamma}X^{\alpha}\,|D^{\alpha\gamma}_{\xi x}(x^{\gamma'}y^{\delta'}\xi^{\alpha'}\eta^{\beta'})| \leqslant Cg$, which implies that $a \in S^{\mu}(\mathbb{R}^n; I)$.

Throughout the remainder of this section $\mu = \log g$. Our main objective in this section is the following lemma:

LEMMA 2.7. *There is a $q \in S^{-\mu}(\mathbb{R}^n; I)$ such that $q_{y\eta}(x, D_x)\, a_{y\eta}(x, D_x)\, u = u$ for all $u \in \mathcal{S}^*$ and all $(y, \eta) \in I$.*

We shall need two preliminary lemmas.

LEMMA 2.8. *There is a $p \in S^{-\mu}(\mathbb{R}^n; I)$ such that for all $N \in \mathbb{N}$, $p \circ a - 1 \in S^{-N\mu}(\mathbb{R}^n; I)$ and $a \circ p - 1 \in S^{-N\mu}(\mathbb{R}^n; I)$. Thus, for each $(y, \eta) \in I$, $p(x, y, D_x, \eta)$ is a two-sided parametrix for $a(x, y, D_x, \eta)$.*

Proof. By Lemma 1.22 there is a C such that $g(x, y, \xi, \eta) \leqslant C\,|a(x, y, \xi, \eta)|$ if $(y, \eta) \in I$ and $|x| + |\xi| \geqslant C$. Thus $a \in S^{\mu}(\mathbb{R}^n; I)$ is "elliptic". The usual construction of a parametrix for an elliptic operator can now be carried through to find the desired $p \in S^{-\mu}(\mathbb{R}^n; I)$. See the first paragraph of the proof of Theorem 7.7 of [2] for details, noting that the appropriate estimates are uniform for $(y, \eta) \in I$.

LEMMA 2.9. *Let $f(y, \eta) = g(0, y, 0, \eta)$. The following are equivalent:*

(2.10) *$r \in S^{-N\mu}(\mathbb{R}^n; I)$, for all $N \in \mathbb{N}$;*

(2.11) *For any $N \in \mathbb{N}$, $\{f(y, \eta)^N r_{y\eta} : (y, \eta) \in I\}$ is bounded in $\mathcal{S}(\mathbb{R}^{2n})$;*

(2.12) *For any $N \in \mathbb{N}$, $\{f(y, \eta)^N r_{y\eta}(x, D_x) : (y, \eta) \in I\}$ is an equicontinuous family of operators from $\mathcal{S}^*(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$.*

Proof. Given α, β, and N, there exist $N' \in \mathbb{N}$, a polynomial $P(x, \xi)$ and a constant $C > 0$ such that

$$\exp(N\mu_{y\eta}(x, \xi))(X^{\alpha}\chi^{\beta})(x, y, \xi, \eta)\,|D^{\alpha\beta}_{\xi x} r(x, y, \xi, \eta)|$$
$$\leqslant Cf(y, \eta)^{N'} P(x, \xi)\,|D^{\alpha\beta}_{\xi x} r(x, y, \xi, \eta)|. \tag{2.13}$$

Conversely, given α, β, N' and P, there exist N and C such that (2.13) holds with the inequality reversed. This proves the equivalence of (2.10) and (2.11). The equivalence of (2.11) and (2.12) follows from the Schwartz kernel theorem and the Banach–Steinhaus theorem (Treves [12]).

Proof of Lemma 2.7. Let $A_{y\eta} = a_{y\eta}(x, D_x)$ and $P_{y\eta} = p_{y\eta}(x, D_x)$, where p has the properties described in Lemma 2.8. If $u \in \mathcal{S}^*$ and $A_{y\eta}u = 0$, then

$u = P_{y\eta}A_{y\eta}u + (I - P_{y\eta}A_{y\eta})\,u \in \mathscr{S}$. By (1.11), $u = 0$. Thus $A_{y\eta}\colon H^1_{y\eta} \to H^0_{y\eta}$ has a left inverse $Q_{y\eta}$, where $H^j_{y\eta}$ is the Sobolev space $H^{j\mu_{y\eta}}_{X_{y\eta}, X^0_{y\eta}}(\mathbb{R}^n)$ as defined in [2]. We shall assume for now that the following holds:

$$Q_{y\eta} \text{ is a bounded two-sided inverse for } A_{y\eta}\,, \text{ all } (y, \eta) \in I. \tag{2.14}$$

We shall later reduce the general case to this case.

Let $R_{y\eta} = r_{y\eta}(x, D_x)$, where $r = 1 - a \circ p$. For each (y, η), $Q_{y\eta} - P_{y\eta} = (I - P_{y\eta}A_{y\eta})\,Q_{y\eta}R_{y\eta} + P_{y\eta}R_{y\eta}$ maps $\mathscr{S}^*$ to $\mathscr{S}$. By the Schwartz kernel theorem

$$(Q_{y\eta} - P_{y\eta})\,u(x) = \int k_{y\eta}(x, x - x')\,u(x')\,dx'$$

for $k_{y\eta} \in \mathscr{S}(\mathbb{R}^{2n})$. Let $r'_{y\eta}(x, \xi) = \int e^{-iz\xi}k(x, z)\,dz$ and $R'_{y\eta} = r'_{y\eta}(x, D_x)$. Then $r'_{y\eta} \in \mathscr{S}(\mathbb{R}^{2n})$, and $R'_{y\eta} = Q_{y\eta}R_{y\eta} = Q_{y\eta} - P_{y\eta}$. Hence $Q_{y\eta}u = q_{y\eta}(x, D_x)\,u$, where $q_{y\eta} = p_{y\eta} + r'_{y\eta}$. For $(y, \eta) \in I$ and $(x, \xi) \in \mathbb{R}^{2n}$ define $q(x, y, \xi, \eta) = q_{y\eta}(x, \xi)$ and $r'(x, y, \xi, \eta) = r'_{y\eta}(x, \xi)$. Since $p \in S^{-\mu}(\mathbb{R}^n; I)$, to prove that $q \in S^{-\mu}(\mathbb{R}^n; I)$ it suffices to show that

$$r' \in S^{-\mu}(\mathbb{R}^n; I). \tag{2.15}$$

Now $f(y, \eta)^N R'_{y\eta} = Q_{y\eta}f(y, \eta)^N R_{y\eta}$. By virtue of Lemmas 2.8 and 2.9 $\{f(y, \eta)^N R_{y\eta}\colon (y, \eta) \in I\}$ is an equicontinuous family of maps from $\mathscr{S}^*(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$. Therefore to prove (2.15) it suffices to show that

$$\{Q_{y\eta}\colon (y, \eta) \in I\} \text{ is an equicontinuous family of maps from } \mathscr{S}(\mathbb{R}^n) \text{ to } \mathscr{S}(\mathbb{R}^n). \tag{2.16}$$

Now by Proposition 3.7 of [10], $\{Q_{y\eta}\colon (y, \eta) \in I\}$ is equicontinuous from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$. Hence $\{Q^*_{y\eta}\colon (y, \eta) \in I\}$ is equicontinuous $(L^2 \to L^2)$. Let $T_{y\eta} = t_{y\eta}(x, D_x)$ where $t = 1 - p \circ a$. Then $\{T^*_{y\eta}\colon (y, \eta) \in I\}$ is equicontinuous $(\mathscr{S}^* \to \mathscr{S})$. It follows that $\{Q^*_{y\eta}T^*_{y\eta}\colon (y, \eta) \in I\}$ is equicontinuous $(\mathscr{S}^* \to \mathscr{S}^*)$, hence $\{T_{y\eta}Q_{y\eta}\colon (y, \eta) \in I\}$ is equicontinuous $(\mathscr{S} \to \mathscr{S})$. The proof of Proposition 3.11 of [2] shows that $\{P_{y\eta}\colon (y, \eta) \in I\}$ is equicontinuous $(\mathscr{S} \to \mathscr{S})$. Since $Q_{y\eta} = T_{y\eta}Q_{y\eta} + P_{y\eta}$, (2.16) follows, thereby proving Lemma 2.7 under the assumption (2.14).

We now drop assumption (2.14). We know that $A_{y\eta}\colon H^1_{y\eta} \to H^0_{y\eta}$ has a left inverse. Let $a^{\#}_{y\eta}$ be the symbol of the formal adjoint $A^*_{y\eta}$ and define $a^{\#}(x, y, \xi, \eta) = a^{\#}_{y\eta}(x, \xi)$. Then $a^{\#}_{y\eta} \circ a_{y\eta}$ is the symbol of $A^*_{y\eta}A_{y\eta}$. It is easily seen that $a^{\#}$ and $a^{\#} \circ a$ are quasi-homogeneous and semi-elliptic with the same weights ρ, ρ' and σ' as a and orders (m, m') and $(2m, 2m')$ respectively. Further $(a^{\#} \circ a)_0 = |\,a_0\,|^2$ where a_0 is given by (1.9).

If $A^*_{y\eta}A_{y\eta}u = 0$ and $u \in \mathscr{S}^*$, then $(A^*_{y\eta}A_{y\eta}u, u) = \|\,A_{y\eta}u\,\|^2 = 0$, so $u = 0$. Since the adjoint of $A^*_{y\eta}A_{y\eta}\colon H^2_{y\eta} \to H^0_{y\eta}$ is $A^*_{y\eta}A_{y\eta}\colon H_{y\eta} \to H^{-2}_{y\eta}$ and these operators are Fredholm ([2], Theorem 7.2), $A^*_{y\eta}A_{y\eta}$ has bounded two sided inverse $Q'_{y\eta}$. By the preceding results $Q'_{y\eta} = q'(x, y, D_x, \eta)$ where $q' \in S^{-2\mu}(\mathbb{R}^n; I)$.

Let $q = q' \circ a^{\#}$, then $q \in S^{-\mu}(\mathbb{R}^n; I)$ by Lemma 2.5. For each $(y, \eta) \in I$ and $j \in \mathbb{R}$, $Q'_{y\eta} A^*_{y\eta}$ is a left inverse for $A_{y\eta}: H^j_{y\eta} \to H^{j-1}_{y\eta}$. Since $\cup_j H^j_{y\eta} = \mathscr{S}^*(\mathbb{R}^n)$ ([2], Lemma 7.5), Lemma 2.7 is proved.

In the next section we shall need to estimate derivatives of q with respect to y and η. Under the additional assumption (2.14) again, this can be done by making use of the following standard formula for differentiating inverses:

$$D_{\eta_i} q = -q \circ D_{\eta_i} a \circ q. \tag{2.17}$$

Similarly for y derivatives. Higher order derivatives can be found by induction.

3. Construction of the Parametrix

We begin by defining local weight vectors $(\Phi, \Psi, \varphi, \psi) = (\Phi_1, \ldots, \Phi_n, \Psi_1, \ldots, \Psi_k; \varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_k)$ on $\mathbb{R}^{n+k}$ and an order $\mu \in O(\Phi, \Psi, \varphi, \psi)$ such that the symbol a of (1.7) is in S^μ. (We refer to [2] for the general definitions. $S^\mu = S^\mu_{\Phi,\Psi,\varphi,\psi}(\mathbb{R}^{n+k})$ is the set of all symbols of order μ. $\mathscr{L}^\mu$ will denote the set of all pseudodifferential operators with symbol in S^μ.) Let g and h be defined by (1.19) and (1.21) and let $\bar{\rho}$ be the number chosen before (2.1). Define

$$r(\xi, \eta) = \sum_1^n |\xi_i|^{m_i/\bar{\rho}} + h(\eta)^M. \tag{3.1}$$

Choose $\bar{\sigma}$ such that

$$\max\{\sigma'_i : i \leqslant s\} < \bar{\sigma} < \min\{m'_i \rho'_i / m'_j : 1 \leqslant i, j \leqslant k\}. \tag{3.2}$$

This is possible because of (1.4). Define $(\Phi, \Psi, \varphi, \psi)$ by

$$\begin{aligned} &\Phi_i^{m_i} = g^{1/(\bar{\rho}-1)} r^{(\bar{\rho}-2)/(\bar{\rho}-1)}, \\ &\varphi_i^{m_i} = g^{1/(\bar{\rho}-1)} r^{-\bar{\rho}/(\bar{\rho}-1)} \qquad \text{for} \quad i = 1, \ldots, n; \\ &\Psi_i^{M} = r^{\bar{\sigma}}, \qquad \psi_i^{M} = r^{-\sigma'_i} \qquad \text{for} \quad i = 1, \ldots, k. \end{aligned} \tag{3.3}$$

Proposition 3.4. *$(\Phi, \Psi, \varphi, \psi)$ is a pair of local weight vectors on $\mathbb{R}^{n+k}$.*

Proof. Let K be any compact subset of $\mathbb{R}^{n+k}$. Let $T = \sup_K |x, y|_{\rho,\sigma'}$. Choose $F \in \mathscr{D}(\mathbb{R})$ such that $F(t) = 1$ if $|t| \leqslant T$. Define

$$g_1(\xi, \eta) = \sum |\xi_i|^{m_i} + \sum |\eta_i|^{m'_i} + 1$$

and

$$\bar{g}(x, y, \xi, \eta) = F(|x, y|_{\rho,\sigma'})\, g(x, y, \xi, \eta) + (1 - F(|x, y|_{\rho,\sigma'}))\, g_1(\xi, \eta).$$

Also, for $i = 1, \ldots, n$ define

$$\bar{\Phi}_i^{m_i} = \bar{g}^{1/(\bar{\rho}-1)} r^{(\bar{\rho}-2)/(\bar{\rho}-1)}, \qquad \bar{\varphi}_i^{m_i} = \bar{g}^{1/(\rho-1)} r^{-\bar{\rho}/(\bar{\rho}-1)}.$$

Then $\bar{\Phi}_i = \Phi_i$ and $\bar{\varphi}_i = \varphi_i$ on K, so to prove the proposition it suffices to show that $(\bar{\Phi}, \Psi, \bar{\varphi}, \psi)$ is a pair of global weight vectors on $\mathbb{R}^{n+k}$.

Note that there is a C such that for all (x, y, ξ, η)

$$r(\xi, \eta) \leqslant \bar{g}(x, y, \xi, \eta) \leqslant Cr(\xi, \eta)^{\bar{\rho}}.$$

These two inequalities imply conditions (2.1), (2.2), (2.4), (2.6) and (2.8) of [2] immediately. Also condition (2.7) of [2] follows by virtue of Proposition 1.20 of [2]. Condition (2.3) of [2] can be verified by showing that there is a C such that for all (x, y, ξ, η)

$$r(\xi + \bar{\Phi}, \eta + \Psi) \leqslant Cr(\xi, \eta) \tag{3.5}$$

and

$$\bar{g}(x + \bar{\varphi}, y + \psi, \xi + \bar{\Phi}, \eta + \Psi) \leqslant C\bar{g}(x, y, \xi, \eta). \tag{3.6}$$

(The functions $\bar{\Phi}$, Ψ, $\bar{\varphi}$ and ψ in the left hand side of (3.5) and (3.6) are evaluated at (x, y, ξ, η).) To prove (3.6) it suffices to show that given compact $K \subset \mathbb{R}^{n+k}$ there is a C such that

$$\begin{aligned} &g(x + \varphi, y + \psi, \xi + \Phi, \eta + \Psi) \\ &\qquad \leqslant Cg(x, y, \xi, \eta) \qquad \text{for} \quad (x, y, \xi, \eta) \in K \times \mathbb{R}^{n+k}. \end{aligned} \tag{3.7}$$

(3.7) can be verified by using definitions (1.18)–(1.21) and (3.3) and considering separately each of the terms obtained from the left side of (3.7). The details are rather messy, but fairly straightforward. They are given in [9], and will be omitted here. Similarly for (3.5) and the remaining condition, (2.5), of [2]. (We note that (3.2) is needed in proving (3.7).)

Note that $g = (\Phi_1\varphi_1)^{m_1(\bar{\rho}-1)/2}\, \Psi^{M/\bar{\sigma}}$, so $\mu = \log g \in O(\Phi, \Psi, \varphi, \psi)$.

PROPOSITION 3.8. *The symbol a of* (1.7) *is in* S^{μ}.

Proof. The proof is similar to that of Lemma 2.6 above, using (3.7) in place of (2.2). We note that we are now using the local theory developed in [2], so the constants may depend on compact subsets in the x, y variables.

We now turn to the problem of constructing a parametrix for $a(x, y, D_x, D_y)$. Choose $\epsilon > 0$ such that $\epsilon < 1$ and $\epsilon < \min\{(\bar{\sigma} - \sigma_i)/2M\}$. For some C_0 yet to be chosen, let

$$\begin{aligned} \Omega_1 &= \{(x, y, \xi, \eta) : |\xi|^2 + |\eta|^2 > C_0 \text{ and } g < 2r^{\epsilon+1}\} \\ \Omega_2 &= \{(x, y, \xi, \eta) : |\xi|^2 + |\eta|^2 > C_0 \text{ and } g > r^{\epsilon+1}\}. \end{aligned} \tag{3.9}$$

If Ω is an open subset of $\mathbb{R}^{n+k} \times \mathbb{R}^{n+k}$ we will write $p \in S^\mu$ on Ω, if given any compact subset K of $\mathbb{R}^{n+k}$ and any multi-indices $(\alpha, \beta, \gamma, \delta)$

$$\sup e^{-\mu}\Phi^\alpha\Psi^\beta\varphi^\gamma\psi^\delta \mid D^{\alpha\beta\gamma\delta}_{\xi\eta xy}p \mid < \infty$$

where the supremum is taken over $(x, y, \xi, \eta) \in \Omega \cap (K \times \mathbb{R}^{n+k})$. The following lemma is analogous to Proposition 11.3 of [2].

LEMMA 3.10. *Let* $\nu = 2\epsilon M^{-1}(\bar{\rho} - 1)^{-1} \log r$ *and* $\mu = \log g$. *If for* $i = 1, 2$ *there exists* $b_i \in S^{-\mu}$ *on* Ω_i *such that* $b_i \circ a - 1 \in S^{-\nu}$ *on* Ω_i, *then* $a(x, y, D_x, D_y)$ *has a left parametrix* $B \in \mathscr{L}^{-\mu}(\mathbb{R}^{n+k})$.

Here $b_i \circ a$ denotes the symbol of the composed operator. By Theorem 4.1 of [2]

$$b_i \circ a = \sum (\alpha!)^{-1} (\beta!)^{-1} (D^{\alpha\beta}_{\xi\eta} b_i)(\partial^{\alpha\beta}_{xy} a). \tag{3.11}$$

This sum is finite since a is a polynomial, so we may take this as the definition of $b_i \circ a$ when b_i is not defined on all of $\mathbb{R}^{n+k} \times \mathbb{R}^{n+k}$.

Proof of Lemma 3.10. We shall find $b_0 \in S^{-\mu}$ on $\Omega_1 \cup \Omega_2$, such that $b_0 \circ a - 1 \in S^{-\nu}$ on $\Omega_1 \cup \Omega_2$. b_0 can then be extended so that $b_0 \in S^{-\mu}(\mathbb{R}^{n+k})$ and $b_0 \circ a - 1 \in S^{-\nu}(\mathbb{R}^{n+k})$. Since $e^\nu \geqslant (1 + |\xi| + |\eta|)^\delta$ for some $\delta > 0$, the lemma will follow from Proposition 11.2 of [2].

Choose $f \in \mathscr{D}(\mathbb{R})$ with $f(t) \equiv 1$ for $|t| \leqslant 4/3$ and $f(t) \equiv 0$ for $|t| \geqslant 5/3$. Let $q_1(x, y, \xi, \eta) = f(r^{-\epsilon-1}g)$ and $q_2 = 1 - q_1$. Then for $i = 1, 2$, $q_i \in S^0(\mathbb{R}^{n+k})$, since the derivatives of q_i are supported where $4/3 \leqslant r^{-\epsilon-1}g \leqslant 5/3$. (We may assume that r and g are smooth by Proposition 3.5 of [2].) Let $e^\lambda = \min\{\Phi_i\varphi_i, \Psi_j\psi_j\}$. Then $\lambda \in O(\Phi, \Psi, \varphi, \psi)$. By (3.3) and the choices of ϵ and ν, we have for all i,

$$\begin{aligned} e^\nu &\leqslant C\Psi_i\psi_i \qquad \text{on} \quad \mathbb{R}^{n+k} \times \mathbb{R}^{n+k} \\ e^\nu &\leqslant Ce^\lambda \qquad \text{on } \Omega_2 . \end{aligned} \tag{3.12}$$

Now define $b_0 = q_1b_1 + q_2b_2$. Since $\operatorname{supp} q_i \cap \{(x, y, \xi, \eta): |\xi|^2 + |\eta|^2 \geqslant C_0\} \subset \Omega_i$, $b_0 \in S^{-\mu}$ on $\Omega_1 \cup \Omega_2$ and $q_i(b_i \circ a - 1) \in S^{-\nu}$ on $\Omega_1 \cup \Omega_2$. Using (3.11) we see that $(q_ib_i) \circ a - q_i(b_i \circ a) \in S^{-\lambda}$ and is supported on the support of the derivatives of q_i, hence on Ω_2. According to (3.12) it follows that

$$b_0 \circ a - 1 = \sum_{i=1,2} q_i(b_i \circ a - 1) + [(q_ib_i) \circ a - q_i(b_i \circ a)] \in S^{-\nu} \quad \text{on } \Omega_1 \cup \Omega_2 .$$

Proof of Theorem 1.12. We shall find b_1 and b_2 so that the hypotheses of Lemma 3.10 are satisfied. The pseudolocal property of pseudodifferential operators will imply that the parametrix is very regular. We construct b_2 first. By Lemma 1.22 there exist C and C' such that $g \leqslant C|a|$ whenever

$g(x, y, \xi, \eta) \geqslant C'$ and $r(\xi, \eta)^{1+\epsilon} \leqslant g(x, y, \xi, \eta)$. Choose C_0 in (3.9) large enough that $|\xi|^2 + |\eta|^2 \geqslant C_0$ implies $g(x, y, \xi, \eta) \geqslant C'$. It follows that $g \leqslant C|a|$ on Ω_2. Therefore, if we let $b_2 = a^{-1}$, then $b_2 \in S^{-\mu}$ on Ω_2 and by (3.12) $b_2 \circ a - 1 \in S^{-\nu}$ on Ω_2. Note that hypothesis (1.10) is not needed in constructing b_2.

We turn to the construction of b_1. It is here that we will use the inverses found in Section 2. First note that since $\epsilon < 1$ and $\bar{\rho} \geqslant 2$, by (3.1) and (3.9) we have

$$r(\xi, \eta) \leqslant Ch(\eta)^M \qquad \text{on } \Omega_1. \tag{3.13}$$

Also we may assume h is smooth, for

$$ch(\eta) \leqslant \left(\sum |\eta_i|^{N/\rho_i'} + 1\right)^{1/N} \leqslant Ch(\eta),$$

and the function in the middle is smooth if N is an appropriate positive integer. By Proposition 3.5 of [2] we may also assume that $r \in S^{\log r}$, hence (3.13) implies that

$$D_\eta^\alpha h(\eta) \leqslant h(\eta)\, \Psi(\xi, \eta)^{-\alpha} \qquad \text{on } \Omega_1. \tag{3.14}$$

Also (3.13) implies that if the C_0 in (3.9) is large enough, then there exists $c > 0$ such that $|\eta| \geqslant c$ on Ω_1; hence there exists $c_1 > 0$ such that $|h(\eta)^{-\rho'}\eta| \geqslant c_1$ on Ω_1. Also by the definition of h, there is a C_1 such that $|h(\eta)^{-\rho'}\eta| \leqslant C_1$. So, if we let $I = \{(y, \eta): c_1 \leqslant |\eta| \leqslant C_1\}$, it follows that

$$(h(\eta)^{\sigma'} y, h(\eta)^{-\rho'}\eta) \in I \qquad \text{if} \quad (x, y, \xi, \eta) \in \Omega_1. \tag{3.15}$$

Let q be the function described in Lemma 2.7. For $(x, y, \xi, \eta) \in \Omega_1$ we now define

$$b_1(x, y, \xi, \eta) = h^{-M} q(h^\rho x, h^{\sigma'} y, h^{-\rho}\xi, h^{-\rho'}\eta),$$

where $h = h(\eta)$. To show that $b_1 \in S^{-\mu}$ on Ω_1, we need the following lemma, which is an immediate consequence of the definitions of g, Φ, φ, X and χ and (3.13). To simplify the notation, for any function p we define the function $\tilde{p}$ by

$$\tilde{p}(x, y, \xi, \eta) = p(h^\rho x, h^{\sigma'} y, h^{-\rho}\xi, h^{-\rho'}\eta) \qquad \text{where} \quad h = h(\eta).$$

LEMMA 3.16. *Let K be a compact subset of $\mathbb{R}^{n+k}$. There is a C such that the following inequalities hold if $(x, y) \in K$ and $(x, y, \xi, \eta) \in \Omega_1$:*

$$h^{-\rho\alpha}(\Phi^\alpha + |\xi^\alpha|) \leqslant C\tilde{X}^\alpha; \qquad \Phi^{-\alpha} h^{\rho\alpha} \leqslant C;$$
$$h^{\rho\alpha}(\varphi^\alpha + |x^\alpha|) \leqslant C\tilde{\chi}^\alpha; \qquad \varphi^{-\alpha} h^{-\rho\alpha} \leqslant C\tilde{X}^\alpha;$$
$$C^{-1} g \leqslant h^M \tilde{g} \leqslant Cg.$$

We now indicate why $b_1 \in S^{-\mu}$ on Ω_1. The estimates

$$\Phi^\alpha \varphi^\beta \mid D^{\alpha\beta}_{\xi x} b_1 \mid \leqslant C_K g^{-1} \qquad \text{on } \Omega_1,$$

follow from (3.15), Lemma 3.16 and the estimates

$$X^\alpha \chi^\beta \mid D^{\alpha\beta}_{\xi x} q \mid \leqslant C g^{-1}$$

established in Lemma 2.7. For the estimates involving y and η derivatives one uses in addition (2.17), (3.14), (1.8) and Proposition 3.8. (There is some checking to do at this point, but it is straightforward. Also (2.17) requires the additional assumption (2.14). If this assumption is dropped, then $q = q' \circ a^{\#}$ as at the end of Section 2. Thus $b_1 = h^{-M}\tilde{q} = h^{-2M}(q')^\sim \circ h^M(a^{\#})^\sim$, where $h^{-2M}(q')^\sim \in S^{-2\mu}$ on Ω_1 and $h^M(a^{\#})^\sim \in S^\mu$. Thus $b_1 \in S^{-\mu}$ on Ω_1.)

It remains to be shown that $b_1 \circ a - 1 \in S^{-\nu}$ on Ω_1. For any $h > 0$, let $V_h: L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be given by $V_h u(x) = u(h^\rho x)$. With $h = h(\eta)$, (1.8) implies that

$$a(x, y, D_x, \eta) = h^M V_h a(x, h^{\sigma'} y, D_x, h^{-\rho'}\eta)\, V_h^{-1},$$

and

$$b_1(x, y, D_x, \eta) = h^{-M} V_h q(x, h^{\sigma'} y, D_x, h^{-\rho'}\eta)\, V_h^{-1},$$

by definition. Therefore, $b_1(x, y, D_x, \eta)\, a(x, y, D_x, \eta) = I$, by Lemma 2.7. Hence

$$\sum (\alpha!)^{-1} D_\xi^\alpha b_1(x, y, \xi, \eta)\, \partial_x^\alpha a(x, y, \xi, \eta) = 1 \qquad \text{on } \Omega_1.$$

It remains to be shown that

$$\sum_{|\beta|>0} (\alpha!)^{-1} (\beta!)^{-1} D^{\alpha\beta}_{\xi\eta} b_1 \partial^{\alpha\beta}_{xy} a \in S^{-\nu} \qquad \text{on } \Omega_1.$$

But $b_1 \in S^{-\mu}$ on Ω_1 and $a \in S^\mu$, so if $|\beta| > 0$, then by (3.12) $| D^{\alpha\beta}_{\xi\eta} b_1 \partial^{\alpha\beta}_{xy} a | \leqslant C\Psi^{-1}\psi^{-1} \leqslant Ce^{-\nu}$ on Ω_1. This completes the proof of Theorem 1.12.

We turn finally to the proof of Theorem 1.13. Hypoellipticity will be established by means of Sobolev space estimates. Let U be an open subset of $\mathbb{R}^N$; Φ and φ local weight vectors on $U \times \mathbb{R}^N$. If $\mu \in O(\Phi, \varphi)$, $\| \;\|_\mu$ will denote some admissible norm on the Sobolev space $H^\mu_{\Phi,\varphi}$ as defined in [2]. If $s \in \mathbb{R}$, $| \;|_s$ will denote the norm on the standard Sobolev space H^s.

Definition. Let $\mu, \lambda \in O(\Phi, \varphi)$ and let $P \in \mathscr{L}^\mu_{\Phi,\varphi}$. P satisfies the estimate E_λ if for every compact $K \subset U$ and every $s \in \mathbb{R}$, there is a C such that for all $u \in \mathscr{D}_K$

$$\| u \|_{\mu+\lambda} \leqslant C(\| Pu \|_\lambda + | u |_{-s}). \tag{3.17}$$

The following theorem is proved in [2].

THEOREM 3.18. *If $P \in \mathscr{L}^{\mu}_{\Phi,\varphi}$ satisfies E_λ for some $\lambda \in O(\Phi, \varphi)$ then P satisfies E_λ for all $\lambda \in O(\Phi, \varphi)$ and P is hypoelliptic.*

Let a be given by (1.7). Again define g and r by (1.21) and (3.1), let Φ, Ψ, φ, ψ be the weight vectors defined by 3.3 and let $\mu = \log g$, $\nu = \log r$. Since $r(\xi, \eta) \geqslant c(1 + |\xi| + |\eta|)^\delta$ for some $\delta > 0$, given $\epsilon_1 > 0$, $\epsilon > 0$, $t > s \geqslant 0$ and compact K, there is a C such that for all $u \in \mathscr{D}_K$,

$$\| u \|_{-\epsilon\nu} \leqslant \epsilon_1 | u |_0 + C | u |_{-s} \tag{3.19}$$

and

$$| u |_{-s} \leqslant \epsilon_1 \| u \|_\mu + C | u |_{-t} . \tag{3.20}$$

Thus to show that $P \in \mathscr{L}^\mu$ satisfies the estimate E_λ it suffices to show that (3.17) holds for some $s \geqslant 0$. Making use of Theorem 6.1 of [2] it follows that given $\epsilon_1 > 0$, $\epsilon > 0$, $s \geqslant 0$ and compact K, there exists C such that for all $u \in \mathscr{D}_K$

$$\| u \|_{\mu-\epsilon\nu} \leqslant \epsilon_1 \| u \|_\mu + C | u |_{-s} . \tag{3.21}$$

Also note that if $P \in \mathscr{L}^\mu$ satisfies E_λ and k is a positive integer, then $P^k \in \mathscr{L}^{k\mu}$ satisfies E_λ .

Proof of Theorem 1.13. Define

$$a_1(x, y, \xi, \eta) = \sum_{A} (a'_{\alpha\beta\gamma\delta}(x, y) - a'_{\alpha\beta\gamma\delta}(0, 0))\, x^\gamma y^\delta \xi^\alpha \eta^\beta.$$

Let A', A and A_1 be the operators with symbols a', a and a_1 respectively. Since A has a left parametrix $B \in \mathscr{L}^{-\mu}$, A satisfies E_λ for all λ. Given $\epsilon > 0$, the neighborhood U_1 of $(0, 0)$ can be chosen small enough that $\| A_1 u \|_0 \leqslant \epsilon \| u \|_\mu$ for all $u \in \mathscr{D}(U_1)$. Hence $A + A_1$ satisfies E_λ on U_1 for $\lambda = 0$, and consequently for all λ.

Now let $q(x, y, \xi, \eta) = -a'_{\alpha\beta\gamma\delta}(x, y)\, x^\gamma y^\delta \xi^\alpha \eta^\beta$ for some $(\alpha, \beta, \gamma, \delta) \in A' - A$, and let $Q = q(x, y, D_x , D_y)$. To finish the proof of Theorem 1.13 it suffices to show that if $P \in \mathscr{L}^\mu$ satisfies E_λ then $P - Q$ satisfies E_λ . If $\rho\alpha + \rho'\beta < M$, then it can be seen that $| x^\gamma y^\delta \xi^\alpha \eta^\beta | \leqslant Cg(x, y, \xi, \eta)^d$ for some $d < 1$. Therefore $q \in S^{\mu-\epsilon\nu}$ for some $\epsilon > 0$ and $P - Q$ satisfies E_λ by (3.21). If $\rho\alpha + \rho'\beta \geqslant M$, then we can choose $\gamma^0 \in \mathbb{Q}_+{}^n$ and $\delta^0 \in \mathbb{Q}_+{}^s$ so that

$$\rho\alpha + \rho'\beta - \rho\gamma^0 - \sigma'\delta^0 = M,$$

and

$$\gamma_i{}^0 < \gamma_i \qquad \text{if} \quad \gamma_i \neq 0, \qquad \delta_i{}^0 < \delta_i \qquad \text{if} \quad \delta_i \neq 0.$$

By Lemma 1.22,

$$| x^{\gamma^0} y^{\delta^0} \xi^\alpha \eta^\beta | \leqslant Cg(x, y, \xi, \eta). \tag{3.22}$$

As in the proof on Proposition 3.8, it follows from (3.22) and (3.7) that $q \in S^{\mu}(U_1)$. Also if k is an integer chosen so that $k(\gamma^0, \delta^0) \in \mathbb{N}^n \times \mathbb{N}^s$, then

$$(x^{\gamma^0} y^{\delta^0} \xi^{\alpha} \eta^{\beta})^k \in S^{k\mu}.$$

Letting q_k be the symbol of Q^k, it follows from the symbolic calculus that

$$q_k(x, y, \xi, \eta) = c(x, y)\, q'(x, y, \xi, \eta) + r(x, y, \xi, \eta)$$

where $q' \in S^{k\mu}(U_1)$, $r \in S^{k\mu - \epsilon\nu}(U_1)$ for some $\epsilon > 0$, $c \in C^{\infty}(U_1)$ and $c(0, 0) = 0$. Therefore, given $\epsilon_1 > 0$, if U_1 is sufficiently small $| Q^k u |_0 \leqslant \epsilon_1 \| u \|_{k\mu} + C \,| u |_0$ for all $u \in \mathscr{D}(U_1)$. Hence $P^k - Q^k$ satisfies E_λ for $\lambda = 0$, since P^k satisfies E_λ. Finally $L = \sum_1^k P^{j-1} Q^{k-j} \in \mathscr{L}^{(k-1)\mu}(U_1)$ and $L(P - Q) - P^k + Q^k \in \mathscr{L}^{k\mu - \epsilon\nu}$. Using Theorem 3.18 again, $P - Q$ satisfies E_λ for all λ, and Theorem 1.13 follows.

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